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Research Article

The Homotopy Perturbation Method to Solve a Wave Equation

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Abstract. In the paper, we discuss applications of *Homotopy Perturbation Method* (HPM) related to wave equations subjected to non-local conditions and the method is applied to two test problems in the paper. The method was introduced by J.-H. He (Homotopy perturbation technique, *Computer Methods in Applied Mechanics and Engineering* **178**(3-4) (1999), 257 – 262) and the solutions are matched against exact solutions as in the literature. The results indicate that the HPM produces accurate solutions and faster converging with less computational effort.

Keywords. Homotopy perturbation method, Wave equation, Non-local conditions, Exact solution

Mathematics Subject Classification (2020). 39B12, 35F25

1. Introduction

The solution of nonlinear equations which are possessing strong non-linearity are normally difficult using analytical methods. This can be easily obtained by using the program such as MATLAB, MATHEMATICA, MAPPLE, or any other open source software. Most of the times, the convergence of the series solution is influenced by the physical parameters in case of semi-analytical methods. Frequently the results obtained will be unsatisfactory, whenever there is a strong nonlinearity. For this type of the problems the solution provides the opportunity to control the convergence region and speed of the series solution as well. In many investigations related to engineering and science, it is observed that the governing equations generate the wave equation. Therefore, this condition has drawn in much thought, and obtaining the solution of the equation has been one of the motivating assignments for mathematicians. For obtaining the solution of the wave equation by analytical methods, it is noticed that the methods are much limited and can be applied in very special cases so they cannot be applied to obtain solution of equations related to numerous realistic situations. Most commonly numerical techniques are applied in order to overcome the difficulties related to size of computational works required and generally the round-off error results into the loss of exactness.

In the year 1999, J.-H. He [7] introduced a *Homotopy Perturbation Method* (HPM) and is again revised by him in 2003. In many physical and engineering phenomena like wave propagation and shallow water waves, it is a well-known fact that these can be modeled by a system of Partial differential equations. From past many years, there is an active research need to obtain accurate and efficient methods to solve a non-linear system of PDEs. The HPM is resulting from Liu's [13] artificial parameter method, and Liao's [12] homotopy analysis method and applicable to linear as well as nonlinear differential equations in producing analytical solutions. For highly nonlinear problems, the analytical solutions can be found easily, this can be regarded as an advantage of HPM. However, Liao [11] supported HPM a special case of the homotopy analysis method. In various areas of nonlinear equations like fluid mechanics and heat transfer HPM has found many applications. The HPM is applied in obtaining solution of various problems related to theory of fluid flow such as Blasius equation in boundary layer theory, He [6]. The researchers (see Barforoushi *et al*. [1], Biazar and Azimi [3], Ezzati and Mousavi [4]) improved the earlier method to solve the nonlinear partial differential equations later on.

For solving one dimensional hyperbolic equation, He [8] deliberated the HPM. Zhang and He [22] obtained solution of the electrostatic potential differential equation. Jin [9] and Ghorbali *et al*. [15] used the HPM in case to solve three dimensional parabolic and hyperbolic equations possessing variable coefficients. To solve nonlinear parabolic and hyperbolic equations the same application was continued by Roozi *et al*. [16].

The nonlocal problems play an important role in real life applications and they used in various field of mathematical physics and in other fields. Karakostas and Tsamatas [10] have studied the boundary value problems with nonlocal conditions. Bellin [2] highlighted the existence of solution for one-dimensional wave equations under same conditions. Ma [14] surveyed the recent developments in nonlocal boundary value problems. Waqas *et al*. [17] have investigated the nonlinear stretched flow of MHD microploar liquid having mixed convection, Joule heating, viscous dissipation, and convective condition. Based on a nonlinear stretched sheet, there is a cause for flow. By employing homotopic procedure, we can achieve analytic solutions. To analyze the convergence of the derived series solutions, numerical values can be presented. In the work of Waqas *et al*. [20], convergence series solutions are established for arising governing set of coupled nonlinear ordinary differential equations by using the homotopy analysis method in order to get the features of different pertinent parameters for temperature and velocity distributions. Recently, Homotopy theory was employed to attain convergent solutions in case of system of nonlinear ordinary differential (see Waqas *et al*. [18]). Waqas *et al*. [21] have been approved the non-Fick's theory of mass species and deliberated the non-Fourier-Fick's heat and mass diffusion theories to study the impact of Burgers' liquid over stretched sheet. Also, the notion of double stratification is involved in the analysis. The governing mathematical model was treated by taking the aid of homotopic procedure. In order to seek the convergent solutions, the created solution equations are confirmed with the assistance of graphs and by numerical calculations. In Waqas *et al*. [19], Homotopy method has employed for simulations of dimensionless nonlinear ordinary differential equations to get the convergent series solutions.

In the first instance tried to describe the HPM in the paper, in continuation method applied to solve the four numerical examples with special reference to nonlocal conditions. Moreover, the solutions of some real life applications are obtained via HPM.

Nomenclature

HPM	:	Homotopy perturbation method
∂	:	the partial derivative
f	:	function of (x,t)
p	:	embedding parameter
R	:	set of all real numbers
w_0	:	initial approximation
r_1, r_2	:	given functions
Ω	:	boundary of the domain of x and t
u, v, w	:	functions of (x,t)
τ	:	limiting value

2. Description of the Method

In HPM, we introduce a new form of technique pertaining to perturbation coupled with the homotopy. In topology two continuous functions from one *topological space* (TS) to another is known as "homo-topic". A homotopy among two continuous functions f as well as g from a TS X to a TS Y is termed as continuous function

$$H : X \times [0,1] \to Y$$

So as

$$H(x,0) = f(x) \text{ and } H(x,1) = g(x), \quad \text{for all } x \in X.$$

The HPM is independent of a small parameter in the equation. In topology homotopy is raised with an embedding parameter $p \in [0,1]$ in consideration, which is as a small parameter.

Let us consider one-dimension wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x,t \in \Omega. \tag{1}$$

Along with initial conditions

$$u(x,0) = r_1(x), \qquad 0 \le x \le \ell, \tag{2}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = r_2(x), \quad 0 \le x \le \ell. \tag{3}$$

The non-homogeneous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = L(t), \quad 0 \le t \le T \tag{4}$$

and nonhomogeneous non-local condition

$$\int_0^L u(x,t)dx = \beta(t), \quad 0 \le t \le T, \tag{5}$$

where f is function of x as well as t, $\Omega = \{(x,t)/0 < x < l, 0 \le t \le T\}$, r_1, r_2 and β are given functions which will satisfy the following:

$$r_1^1(0) = \alpha(0), \quad r_1^2(0) = \alpha'(0),$$

$$\int_0^\ell r_1(x)dx = \beta(0) \text{ and } \int_0^\ell r_1(x)dx = \beta'(0). \tag{6}$$

To solve this non-local problem by the HPM, we first convert this non-local problem into another non-local problem along with homogeneous Neumann condition and a homogeneous non-local condition.

For this, we apply the transformation (see He [7])

$$w(x,t) = u(x,t) - z(x,t), \quad x,t \in \Omega$$

wherein

$$z(x,t) = L(t)\left[x - \frac{\ell}{2}\right] + \frac{\beta(t)}{\ell}$$

there upon

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial^2 z(x,t)}{\partial t^2} \quad \text{with} \quad \frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2}.$$

Therefore, the non-local problem given by equation is converted to the 1-dimensional non-homogeneous equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} + g(x,t)x, \quad t \in \Omega \tag{7}$$

regarding the underlying conditions

$$w(x,0) = 0,\ q_1(x) = 0, \quad 0 \le x \le l, \tag{8}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{r=0} = q_2(x), \quad 0 \le x \le l. \tag{9}$$

The homogeneous Neumann and the homogenous non-local conditions are

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=0} = 0, \quad t \ge 0, \tag{10}$$

$$\int_0^1 w(x,t)dx = 0, \quad t \ge 0, \tag{11}$$

wherein

$$g(x,t)=f(x,t)-\left.\frac{\partial^2 z(x,t)}{\partial t^2}\right|_{x=0}, \quad q(x)=r_2(x)-z(x,0), \quad q_2(x)=r_2(x)-\left.\frac{\partial z(x,t)}{\partial t}\right|_{t=0}.$$

In order to solve the nonlocal problem, we formulate a homotopy $V:\Omega\times[0,1]\to R$ satisfies

$$H(v,p)=\frac{\partial^2 v(x,t,p)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}+p\frac{\partial^2 w_0(x,t)}{\partial t^2}+p\left[\frac{\partial^2 v(x,t,p)}{\partial x^2}-g(x,t)\right]=0, \tag{12}$$

where $p\in[0,1]$, R and w_0 to (7) satisfies the conditions (8)-(11).
By using the equation (12), it follows that

$$H(v,0)=\frac{\partial^2 v(x,t,0)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}=0,$$

$$H(v,1)=\frac{\partial^2 v(x,t,1)}{\partial t^2}-\frac{\partial^2 v(x,t,1)}{\partial x^2}-g(x,t)=0.$$

Now, we approximate the solution of equation as

$$w(x,t,p)=\sum_{i=0}^{\infty}p^i v_i(x,t). \tag{13}$$

Accordingly, the approximated solution of the problem (7) is

$$w(x,t)=\lim_{p\to 1}v(x,t,p)=\sum_{i=0}^{\infty}v_i(x,t). \tag{14}$$

Substitute equation (15) into (12), then

$$H(v,p)=\sum_{i=0}^{\infty}p^i\frac{\partial^2 v_0(x,t)}{\partial t^2}-\frac{\partial^2 v_0(x,t,1)}{\partial t^2}+p\frac{\partial^2 w_0(x,t,)}{\partial t^2}+p\left[-\sum_{t=0}^{\infty}p^i\frac{\partial^2 v_0(x,t)}{\partial x^2}-g(x,t)\right]=0.$$

Then equating like order of p, we have

$$p^0:\frac{\partial^2 v_0(x,t)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}=0 \tag{15}$$

$$p^1:\frac{\partial^2 v(x,t)}{\partial t^2}-\frac{\partial^2 w_0(x,t)}{\partial t^2}-\frac{\partial^2 v_0(x,t)}{\partial t^2}-g(x,t)=0 \tag{16}$$

$$p^j:\frac{\partial^2 v_j(x,t)}{\partial t^2}-\frac{\partial^2 v_{j-1}(x,t)}{\partial x^2}=0, \quad j=2,3,\dots. \tag{17}$$

For simplifying, we choose $v_0(x,t)=w_0(x,t)$. So equation (15) is satisfied automatically.
Consider $w_0(x,t)=q_1(x)+q_2(x)$ thereupon

$$w_0(x,0)=q_1(x), \quad 0\le x\le \ell,$$

$$\left.\frac{\partial w_0(x,0)}{\partial t}\right|_{t=0}=\begin{cases} q_2(x), & 0\le x\le\ell,\\ r_1'(0)-L(0)+r_2'(0)t-L'(0)t, & 0\le t\le T,\\ \beta(0)-\beta(0)+\beta'(0)t-\beta'(0), & t=0.\end{cases}$$

The w_0 satisfied the conditions given in (8)-(11).

Consequently, by putting $t=0$ in (14), we get

$$w(x,0)=\sum_{i=0}^{\infty}v_i(x,0).$$

But

$$v_0(x,0) = q_1(x) \quad \text{and} \quad w(x,0) = q_1(x)$$

hence $v_i(x,0) = 0$, $i = 1,2,3,\dots$.

By using $v_0(x,t) = w_0(x,t) = q_1(x) + q_2(x)t$ in (16) , we get

$$\frac{\partial^2 v_1(x,t)}{\partial t^2} = q_1''(x) + t q_2''(x) + g(x,t).$$

By integrating twice for above differential equation with respect to t with initial conditions

$$v_1(x,0) = 0 \quad \text{and} \quad \left.\frac{\partial v_1(x,t)}{\partial t}\right|_{t=0} = 0.$$

We get

$$v_1(x,t) = \frac{t^2}{2} q_1''(x) + \frac{t^3}{6} q_2''(x) + \int_0^t \int_0^5 g(x,t) dt.$$

By substituting v_1 into equation (17) and by solving the resulting equation with conditions

$$v_2(x,0) = 0 \quad \text{and} \quad \frac{\partial v_2(x,t)}{\partial t}|_{t=0} = 0.$$

One can get $v_2(x,t)$. In a similar manner we obtain $v_i(x,t)$, $i = 3,4,\dots$ by putting $v_i(x,t)$, $i = 3,4,\dots$ in (16), we get the approximation solution w of equations (7).

Therefore, from the equation (6), we have

$$u(x,t) = w(x,t) + z(x,t) = \sum_{i=0}^{\infty} v_i(x,t) + z(x,t), \quad x,t \in \Omega. \tag{18}$$

This is the required solution of original non-local problem (1) and it is clarifying through two numerical problems in subsequent section.

3. Numerical Problems

(1) Consider the homogeneous wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad 0 < x < 1, t > 0 \tag{19}$$

with the initial conditions

$$u(x,0) = \cos(x), \qquad 0 \le x \le \pi, \tag{20}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = -\cos(x), \quad 0 \le x \le \pi. \tag{21}$$

The homogeneous Neumann and non-local conditions are

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0, \quad 0 \le t \le 1, \tag{22}$$

$$\int_0^{\pi} u(x,t) dx = 0, \quad 0 \le t \le 1. \tag{23}$$

For checking the compatibility conditions are satisfied for this nonlocal problem, we use the HPM

$$v_0(x,t) = u_0(x,t) = u(x,0) + \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} t = \cos(x) - t\cos(x).$$

From equation (16) and by the initial conditions

$$\left.\frac{\partial v_1(x,t)}{\partial t}\right|_{t=0} = v_1(x,0) = 0$$

one can have

$$v_1(x,t) = \frac{1}{2!}t^2\cos(x) - \frac{1}{3!}t^3\cos(x).$$

Hence

$$v_0(x,t) + v_1(x,t) = \left[1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3\right]\cos(x).$$

From equation (17) one can get

$$v_2(x,t) = \int_0^t \int_0^\tau \frac{\partial^2 v_1(x,s)}{\partial x^2} ds dT$$

and this implies

$$u(x,t) = \sum_{i=0}^{2} v_i(x,t) = \left[1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5\right]\cos(x)$$

and by proceeding as such one can have

$$u(x,t) = \sum_{i=0}^{\infty} v_i(x,t) = e^{-t}\cos(x). \tag{24}$$

This is the required exact solution and graph of this solution is shown in Figure 1.

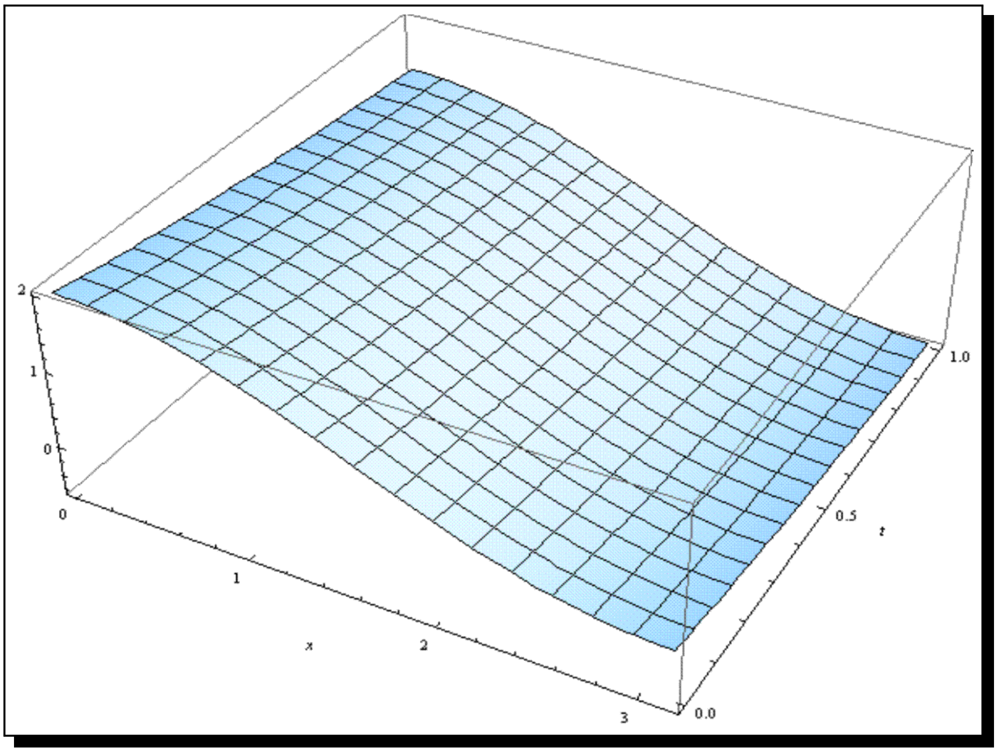

Figure 1

(2) Consider one dimensional in homogeneous wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = -x\sin(t) - 4e^{-2x}, \quad 0 \le x \le \frac{\pi}{2},\ 0 \le t \le 1 \tag{25}$$

subjected to conditions

$$u(x,0) = e^{-2x}, \qquad 0 \le x \le \frac{\pi}{2}, \tag{26}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{t=0} = x, \quad 0 \le x \le \frac{\pi}{2}, \tag{27}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{t=0} = \sin(t) - 2, \quad 0 \le t \le 1, \tag{28}$$

$$\int_0^{\frac{\pi}{2}} u(x,t)dx = \frac{1}{8}\pi^2 \sin(t) - \frac{1}{2}e^{-\pi} + \frac{1}{2}, \quad 0 \le t \le 1. \tag{29}$$

Now it is clear that the conditions (6) are satisfied for this nonlocal problem, we apply the method discussed above. To solve this, use the conversion given by (6).

In this case

$$z(x,t) = (\sin(t) - 2)\left(x - \frac{1}{4}\pi\right) + \frac{1}{\pi}\left[2\left(\frac{1}{8}\pi^2 \sin(t) - \frac{1}{2}\right)e^{-\pi} + 1\right]. \tag{30}$$

Accordingly, the nonlocal problem given by (25) is converted to the 1-d non-homogeneous wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} - 4e^{-2x}, \quad 0 \le t \le 1,\ 0 \le x \le \frac{\pi}{2}$$

with the initial conditions

$$w(x,0) = e^{-2x} + 2x - \frac{\pi}{2} - \frac{1}{\pi}[1 - e^{-\pi}], \quad 0 \le x \le \frac{\pi}{2},$$

$$\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0, \qquad 0 \le x \le \frac{\pi}{2}$$

and the homogeneous Neumann and nonlocal conditions

$$\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0, \quad 0 \le t \le 1,$$

$$\int_0^{\frac{\pi}{2}} w(x,t)dx = 0, \quad 0 \le t \le 1.$$

For getting solution of the problem, employ the same method, let

$$v_0(x,t) = w_0(x,t) = q_1(x) + q_2(x)t = \frac{1}{2\pi}[2\pi e^{-2x} + 4\pi x - \pi^2 + 2e^{-\pi} - 2].$$

From equation $v_1(x,t) = \frac{t^2}{2}q_1''(x) + \frac{t^3}{6}q_2''(x) + \int_0^t \int_0^s g(x,t)dt$ one can have:

$$v_1(x,t) = \frac{t^2}{2}q_1''(x) + \frac{t^3}{6}q_2''(x) + \int_0^t \int_0^s g(x,\tau)d\tau ds = 0.$$

Thus $v_i(x,t) = 0,\ i = 1,2,3,\ldots$

$$\begin{aligned} w(x,t) &= w_0(x,t) \\ &= \frac{1}{2\pi}[2\pi e^{-2x} + 4\pi x - \pi^2 + 2e^{-\pi} - 2] \end{aligned} \tag{31}$$

is the accurate solution of the above nonlocal problem.

Hence it's an exact solution as in the literature

$$u(x,t) = w_0(x,t) + z(x,t) = e^{-2x} + x\sin(t).$$

The graph of solution is shown in Figure 2.

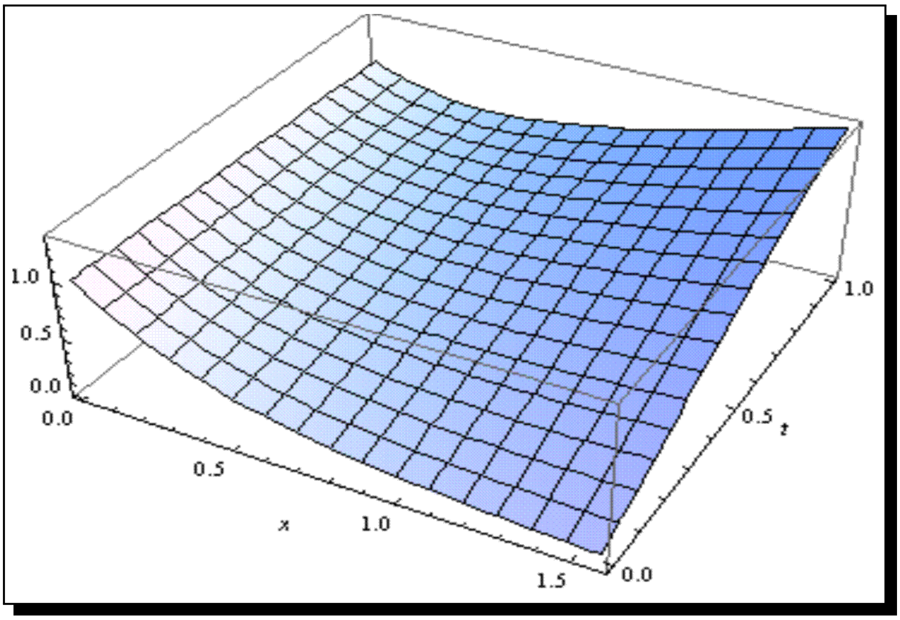

Figure 2

4. Conclusion

This paper demonstrated in revealing the HPM with optimum computational time. This method had employed on two test problems and obtained exact solutions as compared to the existing results. This technique is used in direct way by avoiding difficulties arose in other methods and it does not require any linearization, discretization or assumptions. Finally, it is clear that – it is a promising tool for wave equations with non-local conditions.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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